

THE MOVING PLANE METHOD FOR SINGULAR SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We consider positive solutions to semilinear elliptic problems with singular nonlinearities, under zero Dirichlet boundary condition. We exploit a refined version of the moving plane method to prove symmetry and monotonicity properties of the solutions, under general assumptions on the nonlinearity.

1. INTRODUCTION

In this paper we study symmetry and monotonicity properties of positive solutions to the problem

$$(1.1) \quad \begin{cases} -\Delta u = \frac{1}{u^\gamma} + f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $\gamma > 0$, Ω is a bounded smooth domain of \mathbb{R}^n and $u \in C(\overline{\Omega}) \cap C^2(\Omega)$.

Starting from the pioneering work [14] singular semilinear elliptic equations have been intensively studied, see e.g. [4, 5, 7, 9, 11, 15, 19, 20, 21, 22, 25, 26]. Furthermore, by a simple change of variables, it also follows that the problem is related to equations involving a first order term of the type $\frac{|\nabla u|^2}{u}$. We refer the readers to [1, 6, 16] for related results in this setting.

The main difficulties that we have to face are given by the fact that solutions in general are not in $H_0^1(\Omega)$ and the nonlinearity $\frac{1}{s^\gamma} + f(x, s)$ is not Lipschitz continuous at zero. Note that solutions are not in $H_0^1(\Omega)$ already in the case $f \equiv 0$, see [21]. Therefore, in particular, problem (1.1) has to be understood in the weak distributional meaning with test functions with compact support in Ω , that is

$$(1.2) \quad \int_{\Omega} (\nabla u, \nabla \varphi) dx = \int_{\Omega} \frac{\varphi}{u^\gamma} dx + \int_{\Omega} f(x, u) \varphi dx \quad \forall \varphi \in C_c^1(\Omega).$$

The proof of our symmetry result will be based on the moving plane technique, see [17, 24], as developed and improved in [3]. The crucial point here is the lack of regularity of the solutions near the boundary, that is an obstruction to the use of the test functions technique exploited in [3, 17, 24].

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As we will see, a special role in this issue is plaid by u_0 , the solution to the pure singular problem: $u_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$ and

$$(1.3) \quad \begin{cases} -\Delta u_0 = \frac{1}{u_0^\gamma} & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

The solution u_0 is unique (see [9, 12, 13, 23]) and the existence has been proved in [5, 9]. By the variational characterization provided in [9], it follows that any solution u to problem (1.1) enjoy the decomposition

$$u = u_0 + w \quad \text{for some } w \in H_0^1(\Omega).$$

Such a decomposition has been exploited in [11] (see also the applications in [7, 8, 10]) in order to prove symmetry and monotonicity properties of the solution, via a moving plane type technique applied to w , the $H_0^1(\Omega)$ part of the solution. Since w is not a solution to the problem, such approach required an extra condition on the nonlinearity $f(x, u)$ that, in [11], is assumed to be monotone increasing in the u variable.

The aim of this paper is to remove such a restriction on the nonlinearity and prove symmetry and monotonicity properties of the solution under general assumptions, namely in the case of locally Lipschitz continuous nonlinearities that, more precisely, fulfill

(hp) $f(x, t)$ is a Carathéodory function which is uniformly locally Lipschitz continuous with respect to the second variable. Namely, for any $M > 0$ given, it follows

$$|f(x, t_1) - f(x, t_2)| \leq L_f(M)|t_1 - t_2|, \quad x \in \Omega, \quad t_1, t_2 \in [0, M].$$

Our main result is the following

Theorem 1.1. *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ be a solution to (1.1). Assume that the domain Ω is convex w.r.t. the ν -direction ($\nu \in S^{N-1}$) and symmetric w.r.t. T_0^ν , where*

$$T_0^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = 0\}.$$

With the notation $x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu$, assume that f satisfies (hp), $f(\cdot, t)$ is non decreasing in the $x \cdot \nu$ -direction in the set $\Omega_0^\nu := \Omega \cap \{x \cdot \nu < 0\}$, for all $t \in [0, \infty)$ and

$$f(x, t) = f(x_0^\nu, t) \quad \text{if } x \in \Omega_0 \text{ and } t \in [0, \infty).$$

Then u is symmetric w.r.t. T_0^ν and non-decreasing w.r.t. the ν -direction in Ω_0^ν . In particular, if Ω is a ball centered at the origin of radius $R > 0$, then u is radially symmetric with $\frac{\partial u}{\partial r}(r) < 0$ for $0 < r < R$.

The key point in the proof of Theorem 1.1 is the study of the problem near the boundary. We combine a fine analysis of the behaviour of the solution near the boundary based on comparison arguments that go back to [9], with an improved test functions technique. Let us finally point out that, the monotonicity assumption on f , with respect to the first variable, is necessary for the applicability of the moving plane method. This is well known already in the case of non singular nonlinearities.

2. THE SYMMETRY RESULT

To state the next results we need some notations. Let ν be a direction in \mathbb{R}^N with $|\nu| = 1$. Given a real number λ we set

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\},$$

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\}$$

and

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu,$$

that is the reflection trough the hyperplane T_λ^ν . Moreover we set

$$(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu).$$

Observe that $(\Omega_\lambda^\nu)'$ may be not contained in Ω . Also we take

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu.$$

When $\lambda > a(\nu)$, since Ω_λ^ν is nonempty, we set

$$\Lambda_1(\nu) = \{\lambda : (\Omega_t^\nu)' \subset \Omega \text{ for any } a(\nu) < t \leq \lambda\},$$

and

$$\lambda_1(\nu) = \sup \Lambda_1(\nu).$$

Moreover we set

$$u_\lambda^\nu(x) = u(x_\lambda^\nu),$$

for any $a(\nu) < \lambda \leq \lambda_1(\nu)$. Moreover let us define

$$d(x) = \text{dist}(x, \partial\Omega), \quad \forall x \in \mathcal{I}_\delta(\partial\Omega),$$

whit $\mathcal{I}_\delta(\partial\Omega)$ a neighborhood of radius $\delta > 0$ of $\partial\Omega$, with the *unique nearest point property*, see [2] and the references therein. We start proving the following

Lemma 2.1. *Let u be a solution to (1.1). Then*

$$u(x) \leq C d(x)^{\frac{1}{\gamma+1}} \quad \text{in } \mathcal{I}_\delta(\partial\Omega),$$

for some positive constant $C = C(f, \gamma, \delta, \Omega, \|u\|_{L^\infty(\Omega)})$.

Proof. Since $u \in C(\overline{\Omega})$ and f satisfies (hp), using (1.1), we obtain in the weak distributional meaning

$$-\Delta u \leq \frac{C}{u^\gamma} \quad \text{in } \Omega,$$

for some positive constant $C = C(f, \gamma, \Omega, \|u\|_{L^\infty(\Omega)})$. By [9, Theorem 2.2, Lemma 2.8] it follows that

$$u(x) \leq C u_1(x)^{\frac{1}{\gamma+1}} \quad \text{in } \Omega,$$

where u_1 is the solutions to $-\Delta u_1 = 1$ in Ω with zero Dirichlet boundary condition. Since $u_1 \in C^1(\overline{\Omega})$, the result follows by the mean value theorem. \square

In the following we will denote by $\chi(A)$ the characteristic function of a set A and, with no loss of generality, we will assume that $\nu = e_1$. We have

Proposition 2.2. *For any $\lambda < 0$ we have that*

$$[(u - u_\lambda)^+]^s \cdot \chi(\Omega_\lambda) \in H_0^1(\Omega_\lambda),$$

where $\Omega_\lambda := \{x \in \Omega : x_1 \leq \lambda\}$, provided that

$$s \geq \max\left\{\frac{\gamma + 1}{2}, 1\right\}.$$

Proof. Let $g_\varepsilon(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be locally Lipschitz continuous and such that

$$\begin{cases} g_\varepsilon(t) = 0 & \text{in } [0, \varepsilon], \\ g_\varepsilon(t) = 1 & \text{in } [2\varepsilon, +\infty), \\ g'_\varepsilon(t) \leq \frac{C}{\varepsilon} & \text{in } (\varepsilon, 2\varepsilon). \end{cases}$$

We set

$$\varphi_\varepsilon(x) := \begin{cases} g_\varepsilon(d(x)) & \text{in } \mathcal{I}_\delta(\partial\Omega), \\ 1 & \text{in } \Omega \setminus \mathcal{I}_\delta(\partial\Omega), \end{cases}$$

where it is convenient to choose $\varepsilon > 0$ such that $2\varepsilon < \delta$. We note that

$$(2.4) \quad \text{supp} |\nabla \varphi_\varepsilon| \subseteq \left\{x \in \mathcal{I}_\delta : \varepsilon < d(x) < 2\varepsilon\right\}$$

and almost everywhere

$$|\nabla \varphi_\varepsilon(x)| \leq |g'_\varepsilon(d(x))| |\nabla d(x)| \leq \frac{C}{d(x)}.$$

Let us consider

$$\Psi_\varepsilon = [(u - u_\lambda)^+]^\alpha \varphi_\varepsilon^2 \chi(\Omega_\lambda),$$

with $\alpha > 1$ to be chosen later. By (1.2) we deduce that

$$(2.5) \quad \int_{R_\lambda(\Omega)} (\nabla u_\lambda, \nabla \varphi) dx = \int_{R_\lambda(\Omega)} \frac{\varphi}{(u_\lambda)^\gamma} dx + \int_{R_\lambda(\Omega)} f(x_\lambda, u_\lambda) \varphi dx \quad \forall \varphi \in C_c^1(R_\lambda(\Omega)),$$

as well. By standard density arguments it follows that we can plug Ψ_ε as test function in (1.2) and in (2.5) and then, subtracting, we obtain

$$\begin{aligned} & \alpha \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\alpha-1} \varphi_\varepsilon^2 dx \\ & \leq 2 \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\alpha dx \\ & + \int_{\Omega_\lambda} (u^{-\gamma} - u_\lambda^{-\gamma}) [(u - u_\lambda)^+]^\alpha \varphi_\varepsilon^2 dx \\ & + \int_{\Omega_\lambda} (f(x, u) - f(x_\lambda, u_\lambda)) [(u - u_\lambda)^+]^\alpha \varphi_\varepsilon^2 dx \\ & \leq 2 \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\alpha dx \\ & + \int_{\Omega_\lambda} (f(x, u) - f(x, u_\lambda)) [(u - u_\lambda)^+]^\alpha \varphi_\varepsilon^2 dx, \end{aligned}$$

where we used that $f(\cdot, t)$ is non decreasing in the x_1 -direction in Ω_0 and that $u^{-\gamma} - u_\lambda^{-\gamma} \leq 0$ in the support of $(u - u_\lambda)^+$. Moreover by the assumption (hp)

$$\begin{aligned}
 (2.6) \quad & \alpha \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\alpha-1} \varphi_\varepsilon^2 dx \\
 & \leq 2 \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\alpha dx \\
 & \quad + C(f, \|u\|_{L^\infty(\Omega)}) \int_{\Omega_\lambda} [(u - u_\lambda)^+]^{\alpha+1} \varphi_\varepsilon^2 dx \\
 & \leq 2 \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\alpha dx + C(f, \alpha, \|u\|_{L^\infty(\Omega)}).
 \end{aligned}$$

By weighted Young inequality (2.6) becomes

$$\begin{aligned}
 (2.7) \quad & \frac{\alpha}{2} \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\alpha-1} \varphi_\varepsilon^2 dx \\
 & \leq C(\alpha) \int_{\Omega_\lambda} |\nabla \varphi_\varepsilon|^2 [(u - u_\lambda)^+]^{\alpha+1} \varphi_\varepsilon^2 dx + C(f, \alpha, \|u\|_{L^\infty(\Omega)}).
 \end{aligned}$$

Using Lemma 2.1 and (2.4), we obtain

$$\begin{aligned}
 (2.8) \quad & \int_{\Omega_\lambda} |\nabla \varphi_\varepsilon|^2 [(u - u_\lambda)^+]^{\alpha+1} \varphi_\varepsilon^2 dx = \int_{\Omega_\lambda \cap \text{supp } \nabla \varphi_\varepsilon} |\nabla \varphi_\varepsilon|^2 [(u - u_\lambda)^+]^{\alpha+1} \varphi_\varepsilon^2 dx \\
 & \leq C \int_{\Omega_\lambda \cap \text{supp } \nabla \varphi_\varepsilon} \left(d(x)\right)^{-2} \left(d(x)\right)^{\frac{\alpha+1}{\gamma+1}} dx \leq C \varepsilon^{\frac{\alpha+1}{\gamma+1}-2} \mathcal{L}(\Omega_\lambda \cap \text{supp } \nabla \varphi_\varepsilon)
 \end{aligned}$$

where by $\mathcal{L}(A)$ we denote the Lebesgue measure of a measurable set A . Moreover, since $\Omega_\lambda \cap \text{supp } \nabla \varphi_\varepsilon \subset I_\delta(\partial\Omega)$, then $\mathcal{L}(\Omega_\lambda \cap \text{supp } \nabla \varphi_\varepsilon) \leq C\varepsilon$, for some positive constant $C = C(\Omega)$. Finally from (2.7) and (2.8) we get

$$\int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\alpha-1} \varphi_\varepsilon^2 dx \leq C$$

with $C = C(f, \alpha, \gamma, \delta, \Omega, \|u\|_{L^\infty(\Omega)})$, if $\alpha \geq \gamma$. By Fatou's Lemma we obtain

$$[(u - u_\lambda)^+]^{\frac{\alpha+1}{2}} \in H_0^1(\Omega_\lambda), \quad \text{if } \alpha \geq \gamma.$$

□

Proof of Theorem 1.1. Using the same notations of the proof of Proposition 2.2 and arguing as above, we consider

$$\Psi_\varepsilon = [(u - u_\lambda)^+]^\beta \varphi_\varepsilon^2 \chi(\Omega_\lambda),$$

with

$$(2.9) \quad \beta > \max \left\{ 1, \gamma, (\gamma + 1)/2 \right\}.$$

By density arguments we plug Ψ_ε as test function in (1.2) and in (2.5) and then, subtracting, we get that

$$\begin{aligned}
(2.10) \quad & \beta \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\beta-1} \varphi_\varepsilon^2 dx \\
& \leq 2 \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\beta dx \\
& \quad + \int_{\Omega_\lambda} (u^{-\gamma} - u_\lambda^{-\gamma}) [(u - u_\lambda)^+]^\alpha \varphi_\varepsilon^2 dx \\
& \quad + \int_{\Omega_\lambda} (f(x, u) - f(x, u_\lambda)) [(u - u_\lambda)^+]^\beta \varphi_\varepsilon^2 dx, \\
& \leq 2 \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\beta dx \\
& \quad + C(f, \|u\|_{L^\infty(\Omega)}) \int_{\Omega_\lambda} [(u - u_\lambda)^+]^{\beta+1} \varphi_\varepsilon^2 dx.
\end{aligned}$$

We estimate the first term on the right-hand side of the last line of (2.10) as follows. Using Hölder inequality and then Proposition (2.2) we obtain

$$\begin{aligned}
& \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\beta dx \\
& \leq C(\gamma) \left(\int_{\Omega_\lambda} \left| \nabla [(u - u_\lambda)^+]^{\frac{\gamma+1}{2}} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_\lambda} |\nabla \varphi_\varepsilon|^2 [(u - u_\lambda)^+]^{2\beta-(\gamma-1)} dx \right)^{\frac{1}{2}} \\
& \leq C \left(\int_{\Omega_\lambda} |\nabla \varphi_\varepsilon|^2 [(u - u_\lambda)^+]^{2\beta-(\gamma-1)} dx \right)^{\frac{1}{2}},
\end{aligned}$$

with $C = C(f, \gamma, \delta, \Omega, \|u\|_{L^\infty(\Omega)})$. Using (2.4) again we infer that

$$\int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\beta dx \leq C \varepsilon^{\frac{2(\beta-\gamma)}{\gamma+1}}$$

and then

$$(2.11) \quad \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+| |\nabla \varphi_\varepsilon| \varphi_\varepsilon [(u - u_\lambda)^+]^\beta dx = o(1), \quad \text{as } \varepsilon \rightarrow 0,$$

since, by (2.9), $2(\beta - \gamma)/(\gamma + 1) > 0$. Then by (2.10) and (2.11), passing to the limit, we deduce that

$$(2.12) \quad \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\beta-1} dx \leq C \int_{\Omega_\lambda} [(u - u_\lambda)^+]^{\beta+1} dx,$$

with $C = C(\beta, \gamma, f, \|u\|_{L^\infty(\Omega)})$ a positive constant. As a consequence of Proposition 2.2, recalling (2.9), we can apply Poincaré inequality in the r.h.s of (2.12) to deduce that

$$\begin{aligned}
(2.13) \quad & \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\beta-1} dx \leq C \int_{\Omega_\lambda} \left([(u - u_\lambda)^+]^{\frac{\beta+1}{2}} \right)^2 dx \\
& \leq C \cdot C_P(\Omega_\lambda) \int_{\Omega_\lambda} |\nabla(u - u_\lambda)^+|^2 [(u - u_\lambda)^+]^{\beta-1} dx,
\end{aligned}$$

were $C_P(\Omega_\lambda) \rightarrow 0$ as $\mathcal{L}(\Omega_\lambda) \rightarrow 0$. Thus, there exists $\delta = \delta(n, \beta, \gamma, f, \|u\|_{L^\infty(\Omega)})$ such that if

$$(2.14) \quad \mathcal{L}(\Omega_\lambda) \leq \delta,$$

then $C \cdot C_P(\Omega_\lambda) < 1$ in (2.13). This implies that

$$(2.15) \quad (u - u_\lambda)^+ = 0 \quad \text{in } \Omega_\lambda,$$

namely $u \leq u_\lambda$ in Ω_λ .

Claim: there exists $\bar{\mu} > 0$ small such that

$$(2.16) \quad u < u_\lambda \quad \text{in } \Omega_\lambda,$$

for any $a(e_1) < \lambda \leq a(e_1) + \bar{\mu}$.

In fact we can fix $\bar{\mu} > 0$ small so that (2.15) holds and provides that

$$u \leq u_\lambda \quad \text{in } \Omega_\lambda,$$

for any $a(e_1) < \lambda \leq a(e_1) + \bar{\mu}$. Therefore we only need to prove the strict inequality. To prove this assume by contradiction that, for some λ , with $a(e_1) < \lambda \leq a(e_1) + \bar{\mu}$, there exists a point $x_0 \in \Omega_\lambda$ such that $u(x_0) = u_\lambda(x_0)$. Then let $r = r(x_0) > 0$ be such that $B_r(x_0) \subset\subset \Omega'_\lambda$. We have, in the classical sense (since $u \in C^2(\Omega)$),

$$(2.17) \quad -\Delta(u - u_\lambda) = \left(\frac{1}{u^\gamma} - \frac{1}{u_\lambda^\gamma} \right) + \left(f(x, u) - f(x_\lambda, u_\lambda) \right) \quad \text{in } B_r(x_0).$$

From (1.1), we deduce that there exists a positive constant $C = C(r, \lambda)$ such that

$$\min_{x \in B_r(x_0)} \{u(x), u_\lambda(x)\} \geq C > 0.$$

Then (using the assumption *(hp)* as well) we can estimate the r.h.s to (2.17) as

$$\left| \left(\frac{1}{u^\gamma} - \frac{1}{u_\lambda^\gamma} \right) + \left(f(x, u) - f(x_\lambda, u_\lambda) \right) \right| \leq C|u - u_\lambda| \quad \text{in } B_r(x_0),$$

with $C = C(f, r, \lambda, \|u\|_{L^\infty(\Omega)})$. Hence we find $\Lambda > 0$ such that, from (2.17), we obtain

$$-\Delta(u - u_\lambda) + \Lambda(u - u_\lambda) \geq 0 \quad \text{in } B_r(x_0)$$

and we are in position to exploit the strong maximum principle [18] to deduce that $u \equiv u_\lambda$ in $B_r(x_0)$. By a covering argument it would follow that $u \equiv u_\lambda$ in Ω_λ providing a contradiction with the Dirichlet condition and thus proving the claim.

To proceed further we set

$$\Lambda_0 = \{\lambda > a(e_1) : u < u_t \text{ in } \Omega_t \text{ for all } t \in (a(e_1), \lambda]\},$$

which is not empty thanks to (2.15). Also set

$$\lambda_0 = \sup \Lambda_0.$$

We have to show that actually $\lambda_0 = \lambda_1(e_1) = 0$. Assume otherwise that $\lambda_0 < 0$ and note that, by continuity, we obtain that $u \leq u_{\lambda_0}$ in Ω_{λ_0} . Repeating verbatim the argument used in the proof of the previous claim, we deduce that $u < u_{\lambda_0}$ in Ω_{λ_0} unless $u = u_{\lambda_0}$ in Ω_{λ_0} . But, as above, because of the zero Dirichlet boundary conditions and since $u > 0$ in the interior of the domain, the case $u \equiv u_{\lambda_0}$ in Ω_{λ_0} is not possible if $\lambda_0 < 0$. Thus $u < u_{\lambda_0}$ in Ω_{λ_0} .

Now we fix a compact set $\mathcal{K} \subset \Omega_{\lambda_0}$ so that $\mathcal{L}(\Omega_{\lambda_0} \setminus \mathcal{K}) \leq \frac{\delta}{2}$, with δ given by (2.14). By compactness we find $\sigma = \sigma(\mathcal{K}) > 0$ such that

$$u_{\lambda_0} - u \geq 2\sigma > 0 \quad \text{in } \mathcal{K}.$$

Take now $\bar{\varepsilon} > 0$ sufficiently small so that $\lambda_0 + \bar{\varepsilon} < \lambda_1(\nu)$ and for any $0 < \varepsilon \leq \bar{\varepsilon}$

- a) $u_{\lambda_0+\varepsilon} - u \geq \sigma > 0$ in \mathcal{K} ,
- b) $\mathcal{L}(\Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}) \leq \delta$.

Taking into account a) it is now easy to check that, for any $0 < \varepsilon \leq \bar{\varepsilon}$, we have that $u \leq u_{\lambda_0+\varepsilon}$ on the boundary of $\Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}$. Now we argue as above but considering the test function

$$\Psi_\varepsilon = [(u - u_{\lambda_0+\varepsilon})^+]^\alpha \varphi_\varepsilon^2 \chi(\Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}).$$

Following verbatim the arguments from equation (2.9) to equation (2.15), since b) holds, we obtain

$$u \leq u_{\lambda_0+\varepsilon} \quad \text{in } \Omega_{\lambda_0+\varepsilon} \setminus \mathcal{K}.$$

Thus $u \leq u_{\lambda_0+\varepsilon}$ in $\Omega_{\lambda_0+\varepsilon}$. We get a contradiction with the definition of λ_0 and conclude that actually $\lambda_0 = \lambda_1(\nu)$. Then it follows that

$$u(x) \leq u_0(x) \quad \text{for } x \in \Omega_0^{e_1}.$$

In the same way, performing the moving plane method in the direction $-e_1$ we obtain

$$u(x) \geq u_0(x) \quad \text{for } x \in \Omega_0^{e_1},$$

that is, u is symmetric w.r.t. $T_0^{e_1}$ and non-decreasing w.r.t. the e_1 -direction in $\Omega_0^{e_1}$.

Finally, if Ω is a ball of radius $R > 0$, repeating the argument for any direction, it follows that u is radially symmetric. The fact that $\frac{\partial u}{\partial r}(r) < 0$ for $0 < r < R$, follows by the Hopf's Lemma.

□

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